

ITP-SB-93-58
September, 1993

General Determination of Phases in Quark Mass Matrices

Alexander Kusenko¹ and Robert Shrock²

Institute for Theoretical Physics
State University of New York
Stony Brook, N. Y. 11794-3840

Abstract

We construct new invariants and give several theorems which determine in general (i) the number of physically meaningful phases in quark mass matrices and (ii) which elements of these matrices can be rendered real by rephasings. We illustrate our results with simple models.

¹email: sasha@max.physics.sunysb.edu

²email: shrock@max.physics.sunysb.edu

Understanding fermion masses and quark mixing remains one of the most important outstanding problems in particle physics. In an effort to gain insight into this problem, many studies of simple models of quark mass matrices have been carried out over the years. The phases in these mass matrices play an essential role in the Kobayashi-Maskawa (KM) mechanism [1] for CP violation [2]. A given model is characterized by the number of parameters (amplitudes and phases) which specify the quark mass matrices. Thus, a very important problem is to determine, for any model, how many physically meaningful complex phases (i.e. phases $\neq 0$ or π) occur in the quark mass matrices and which elements of these matrices can be made real by rephasings of quark fields. Surprisingly, there is no general solution to this question in the literature. In this Letter we shall present a general solution and apply our results to several models.

The quark mass terms can be written in terms of the $SU(3) \times SU(2) \times U(1)$ fields as

$$-\mathcal{L}_m = \sum_{j,k=1}^3 \left[v_u \bar{Q}_{1jL} Y_{jk}^{(u)} u_{kR} + v_d \bar{Q}_{2jL} Y_{jk}^{(d)} d_{kR} \right] + h.c. \quad (1)$$

where j, k denote generation indices, $Q_{1L} = \begin{pmatrix} u \\ d \end{pmatrix}_L$, $Q_{2L} = \begin{pmatrix} c \\ s \end{pmatrix}_L$, $Q_{3L} = \begin{pmatrix} t \\ b \end{pmatrix}_L$; the first subscript on $Q_{a,jL}$ is the $SU(2)$ index; $u_{1R} = u_R$, $u_{2R} = c_R$, and so forth for d_{kR} [3]. Here we assume three generations of standard model fermions. $Y^{(u)}$ and $Y^{(d)}$ are the Yukawa matrices in the up and down quark sectors whose diagonalization yields the mass eigenstates u_{jm} and d_{jm} . The v_u and v_d are mass parameters (Higgs vacuum expectation values in theories with fundamental Higgs).

To count the number of physically meaningful phases, we rephase the fermion fields so as to remove all possible phases in the $Y^{(f)}$. We consider the quark sector first. Here, one can perform the rephasings defined by

$$Q_{jL} = e^{-i\alpha_j} Q'_{jL} \quad (2)$$

$$u_{jR} = e^{i\beta_j^{(u)}} u'_{jR} \quad (3)$$

$$d_{jR} = e^{i\beta_j^{(d)}} d'_{jR} \quad (4)$$

for $j = 1, 2, 3$. In terms of the primed (rephased) fermion fields, the Yukawa matrices have

elements

$$Y_{jk}^{(f)\prime} = e^{i(\alpha_j + \beta_k^{(f)})} Y_{jk}^{(f)} \quad (5)$$

for $f = u, d$. Thus, if $Y^{(f)}$ has N_f nonzero, and, in general, complex elements, then the N_f equations for making these elements real are

$$\alpha_j + \beta_k^{(f)} = -\arg(Y_{jk}^{(f)}) + \eta_{jk}^{(f)}\pi \quad (6)$$

for $f = u, d$, where the set $\{jk\}$ runs over each of these nonzero elements, and $\eta_{jk}^{(f)} = 0$ or 1 [4]. Let us define the 9-dimensional vector of fermion field phases

$$v = (\{\alpha_i\}, \{\beta_i^{(u)}\}, \{\beta_i^{(d)}\})^T \quad (7)$$

where $\{\alpha_i\} \equiv \{\alpha_1, \alpha_2, \alpha_3\}$, etc., and

$$w = (\{\arg(Y^{(u)}) + \eta_{jk}^{(u)}\pi\}, \{\{\arg(Y^{(d)}) + \eta_{jk}^{(d)}\}\})^T \quad (8)$$

of dimension equal to the number of rephasing equations $N_{eq} = N_u + N_d$. We can then write (6) for $f = u, d$ as

$$Tv = w \quad (9)$$

which defines the N_{eq} -row by 9-column matrix T .

We first note that $\text{rank}(T) \leq 8$. This is proved by ruling out the only other possibility, i.e. $\text{rank}(T) = 9$. The reason that $\text{rank}(T)$ cannot have its apparently maximal value is that one overall rephasing has no effect on the Yukawa interaction, namely the U(1) generated by (2)-(4) with $-\alpha_i = \beta_j^{(u)} = \beta_k^{(d)}$ for all i, j, k .

Our first main theorem is: The number of unremovable phases N_p in $Y^{(u)}$ and $Y^{(d)}$ is

$$N_p = N_{eq} - \text{rank}(T) \quad (10)$$

This is proved as follows. Let $\text{rank}(T) = r_T$. Then one can delete $N_{eq} - r_T$ rows from the matrix A , i.e. not attempt to remove the phases from the corresponding elements of the $Y^{(f)}$, $f = u, d$. For the remaining r_T equations, one moves a subset of $9 - (N_{eq} - r_T)$ phases in v to the right-hand side of (6), thus including them in a redefined \bar{w} . This yields a set

of r_T linear equations in r_T unknown phases, denoted \bar{v} . We write this as $\bar{T}\bar{v} = \bar{w}$. Since by construction $\text{rank}(\bar{T}) = r_T$, \bar{T} is invertible, so that one can now solve for the r_T fermion rephasings in \bar{v} which render r_T of the N_{eq} complex elements real. Hence there are $N_{eq} - r_T$ remaining phases in the $Y^{(f)}$, as claimed. \square .

Some comments are in order. First, as is clear from our proof, the result (10) does not depend on whether or not $Y_{jk}^{(f)} = Y_{kj}^{(f)}$, and hence making Y_f (complex) symmetric does not, in general, result in any reduction in N_p . Second, if one of the unremovable phases is put in a given off-diagonal $Y_{pq}^{(f)}$, one may wish to modify the $qp'th$ equation to read

$$\alpha_q + \beta_p^{(f)} = -\arg(Y_{qp}^{(f)}) - \arg(Y_{pq}^{(f)}) \quad (11)$$

For example, in a model where $|Y_{pq}^{(f)}| = |Y_{qp}^{(f)}|$, this would yield $Y_{pq}^{(f)*} = Y_{qp}^{(f)}$ for this pair pq . The modification in (11) has no effect on the counting of phases.

A fundamental question concerns which elements of $Y^{(u)}$ and $Y^{(d)}$ can be made real by fermion rephasings. This is connected with the issue of which rows are to be removed from T to obtain \bar{T} , i.e. which nonzero elements of the $Y^{(u)}$ and $Y^{(d)}$ are left complex. We present two more theorems which answer this question. The general method is to construct all independent complex rephasing-invariant (wrt. (2)-(4)) products of elements of the $Y^{(f)}$, $f = u, d$. These must involve an even number of such elements. Since, by construction, these have arguments $\neq 0, \pi$, each one implies a constraint which is that the set of $2n$ elements which comprise it cannot be made simultaneously real by any fermion rephasings. We thus construct a set of invariants depending on the up and down quark sectors individually:

$$P_{2n;j_1k_1,\dots,j_nk_n;\sigma_L}^{(f)} = \prod_{a=1}^n Y_{j_ak_a}^{(f)} Y_{\sigma_L(j_a)k_a}^{(f)*} \quad (12)$$

where $f = u, d$, and σ_L is an element of the permutation group S_n . Secondly, we construct a set of invariants connecting the up and down quark sectors:

$$Q_{2n;\{j\},\{k\},\{m\};\sigma_L,\sigma_u;\sigma_d}^{(s,t)} = \left(\prod_{a=1}^s Y_{j_ak_a}^{(u)} \right) \left(\prod_{b=1}^t Y_{j_{s+b}m_b}^{(d)} \right) \left(\prod_{c=1}^s Y_{\sigma_L(j_c)\sigma_u(k_c)}^{(u)*} \right) \left(\prod_{e=1}^t Y_{\sigma_L(j_{s+e})\sigma_d(m_e)}^{(d)*} \right) \quad (13)$$

where $s, t \geq 1$, $s + t = n$, $\sigma_L \in S_n$, $\sigma_u \in S_s$, and $\sigma_d \in S_t$. At quartic order, $2n = 4$, if σ in (12) equals the transposition τ , we obtain the complex invariants

$$P_{4;j_1k_1,j_2k_2;\tau}^{(f)} \equiv P_{j_1k_1,j_2k_2}^{(f)} = Y_{j_1k_1}^{(f)} Y_{j_2k_2}^{(f)} Y_{j_2k_1}^{(f)*} Y_{j_1k_2}^{(f)*} \quad (14)$$

for $f = u, d$. At this order, there is only one Q -type complex invariant; this has $s = t = 1$, $\sigma_L = \tau$, and we denote it simply as

$$Q_{j_1 k_1, j_2 m_1} = Y_{j_1 k_1}^{(u)} Y_{j_2 m_1}^{(d)} Y_{j_2 k_1}^{(u)*} Y_{j_1 m_1}^{(d)*} \quad (15)$$

Note that $P_{j_1 k_1, j_2 k_2}^{(f)} = P_{j_2 k_2, j_1 k_1}^{(f)}$, $P_{j_1 k_1, j_2 k_2}^{(f)*} = P_{j_1 k_2, j_2 k_1}^{(f)*}$, and $Q_{j_1 k_1, j_2 m_1} = Q_{j_2 k_1, j_1 m_1}^*$.

At order $2n = 6$, we find one independent P -type complex invariant for each $f = u, d$, and two independent Q -type complex invariants, which we denote in a simple notation as

$$P_{j_1 k_1, j_2 k_2, j_3 k_3}^{(f)} = Y_{j_1 k_1}^{(f)} Y_{j_2 k_2}^{(f)} Y_{j_3 k_3}^{(f)} Y_{j_2 k_1}^{(f)*} Y_{j_3 k_2}^{(f)*} Y_{j_1 k_3}^{(f)*} \quad (16)$$

$$Q_{j_1 k_1, j_2 k_2, j_3 m_1}^{(fff')} = Y_{j_1 k_1}^{(f)} Y_{j_2 k_2}^{(f)} Y_{j_3 m_1}^{(f')} Y_{j_1 k_2}^{(f)*} Y_{j_3 k_1}^{(f)*} Y_{j_2 m_1}^{(f')*} \quad (17)$$

where $(fff') = (uud)$ or (ddu) . We find that there are no new constraints from any invariant of order ≥ 8 . Note that $P_{6; j_1 k_1, j_2 k_2, j_3 k_3}^{(f)} = P_{j_3 k_2, j_2 k_1, j_1 k_3}^{(f)*}$, $Q_{6; j_1 k_1, j_2 k_2, j_3 m_1}^{(fff')} = Q_{6; j_1 k_2, j_3 k_1, j_2 m_1}^{(fff')*}$.

Our theorems are as follows: For a given model, construct the maximal set of independent complex invariants of lowest order, whose arguments (phases) are linearly independent. Denote the number of these by N_{ia} . Then (a) each of these invariants implies a constraint that the elements contained within it cannot all be made simultaneously real; (b) this constitutes the complete set of constraints on which elements of $Y^{(u)}$ and $Y^{(d)}$ can be made simultaneously real; and hence (c) $N_p = N_{ia}$. Further, (d) there are no new constraints from any invariant of order ≥ 8 . Details of proofs are given in [5]. Usually, the lowest-order nonvanishing complex invariants are quartic (an exception is given below).

Note that a set of N_{inv} independent complex invariants of lowest order, say, will have arguments which are not in general linearly independent, so $N_{inv} \geq N_{ia}$. For each complex invariant of a given order, X , $\arg(X) = \sum_{f=u,d} \sum_{j,k} c_{j,k}^{(f)} \arg(Y^{(f)})_{jk}$, where the sum is over the N_{eq} complex elements of $Y^{(u)}$ and $Y^{(d)}$. These equations can be written as $Z\xi = w$, where ξ is the N_{inv} -dimensional vector $\xi = (\arg(X_1), \dots, \arg(X_{N_{inv}}))$ and Z is an N_{inv} -row by N_{eq} -column matrix. Then $\text{rank}(Z) = N_{ia}$.

We illustrate our general theorems with some models [6]. The first is a generalization of

models studied in Refs. [10] and [11]:

$$Y^{(u)} = \begin{pmatrix} 0 & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix} \quad (18)$$

$$Y^{(d)} = \begin{pmatrix} 0 & B_{12} & 0 \\ B_{21} & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{pmatrix} \quad (19)$$

where each of the elements is, in general, complex. Calculating the 10×9 matrix, T , we find $\text{rank}(T) = 8$ so from (10), there are $N_p = 2$ unremovable phases in the Y_f . Correspondingly, there are $N_{inv} = N_{ia} = 2$ complex invariants (with independent arguments): First, $P_{22,33}^{(d)} = B_{22}B_{33}B_{32}^*B_{23}^*$ is nonzero, and since, in general, $\arg(P_{22,33}^{(d)}) \neq 0, \pi$, it follows that (i) at least one of these phases must reside among the set $S_1 = \{B_{22}, B_{23}, B_{32}, B_{33}\}$ [12], and (ii) the 2×2 submatrix in $Y^{(d)}$ formed by S_1 (and hence also $Y^{(d)}$ itself) cannot be made hermitian [13]. Second, $Q_{12,22} = A_{12}B_{22}A_{22}^*B_{12}^*$ is nonzero, and in general $\arg(Q_{12,22}) \neq 0, \pi$. Hence, (iii) if one chooses $Y^{(u)}$ real, then it is not possible to make B_{12} and B_{22} both real. Conditions (i)-(iii) are a complete set of rephasing constraints on the $Y^{(f)}$. These constraints allow both phases to be put in $Y^{(d)}$ and both to be put in the set S_1 . If one chooses to make B_{22} and B_{12} real, then one cannot make $Y^{(u)}$ real, and must assign one phase to A_{12} or A_{22} and the second to B_{23} , B_{32} or B_{33} . In accord with our theorem, although this model has one complex 6'th order invariant, $Q_{32,23,12}^{(ddu)} = B_{32}B_{23}A_{12}B_{33}^*B_{12}^*A_{22}^*$, its argument can be expressed in terms of quartic invariant arguments: $\arg(Q_{32,23,12}^{(ddu)}) = \arg(Q_{12,22}) - \arg(P_{22,33}^{(d)})$.

A second model is given by

$$Y^{(u)} = \begin{pmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix} \quad (20)$$

(Fritzsch form [14]) and $Y^{(d)}$ as in (19) [10]. We find that the corresponding 11×9 matrix T has $\text{rank}(T) = 8$, so $N_p = 3$. There are $N_{inv} = 4$ nonzero independent complex invariants: $P_{22,33}^{(d)}$ as in model 1, together with $Q_{12,32} = A_{12}B_{32}A_{32}^*B_{12}^*$, $Q_{23,32} = A_{23}B_{32}A_{33}^*B_{22}^*$, and $Q_{23,33} = A_{23}B_{33}A_{33}^*B_{23}^*$. These have only $N_{ia} = 3$ independent arguments, since $\arg(P_{22,33}^{(d)}) + \arg(Q_{23,32}) - \arg(Q_{23,33}) = 0$. The constraints on rephasing from $P_{22,33}^{(d)}$ are

(i) and (ii) as above; and (iii) none of the sets $\{A_{12}, A_{32}, B_{12}, B_{32}\}$, $\{A_{23}, A_{33}, B_{22}, B_{32}\}$, and $\{A_{23}, A_{33}, B_{23}, B_{33}\}$ can be made simultaneously real. In particular, if $Y^{(u)}$ is made real, then none of the sets $\{B_{12}, B_{32}\}$, $\{B_{32}, B_{22}\}$, and $\{B_{23}, B_{33}\}$ can be made simultaneously real.

A third model is given by

$$Y^{(u)} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{31} & 0 & A_{33} \end{pmatrix} \quad (21)$$

with $Y^{(d)}$ as in (18) (this generalizes models considered in [15] and [10]). Here T is 10×9 with $\text{rank}(T) = 8$, so $N_p = 2$. We find $N_{inv} = N_{ia} = 2$ complex quartic invariants, $P_{22,33}^{(d)}$ and $Q_{13,32}$. The constraint from $P_{22,33}^{(d)}$ is given in (i), (ii) above; the constraint from $Q_{13,32}$ is that the set $\{A_{13}, A_{31}, B_{12}, B_{32}\}$ cannot be made simultaneously real.

An example of a model for which the lowest order (nonzero) complex invariant occurs at 6'th order is defined by $Y^{(u)}$ as given by (20) and $Y^{(d)}$ by (18) with $A_{jk} \rightarrow B_{jk}$ [16]. Here, T is 9×9 with rank 8, whence $N_p = 1$. The $N_{inv} = N_{ia} = 1$ 6'th order invariant is $Q_{32,23,12}^{(uud)} = A_{32}A_{23}B_{12}A_{12}^*A_{33}^*B_{22}^*$. This yields the constraint that if $Y^{(u)}$ is made real, then $\{B_{12}, B_{22}\}$ cannot be made real, and if $Y^{(d)}$ is made real, then $\{A_{12}, A_{23}, A_{32}, A_{33}\}$ cannot all be made real. Further details and applications will be given elsewhere [5].

This research was supported in part by NSF grant PHY-93-09888. We thank L. Lavoura for a discussion on the quartic P invariant.

References

- [1] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).
- [2] We consider here the physically relevant case of three generations of standard model fermions; the quark mixing matrix then involves one CP-violating phase, δ . In general, unremovable phase(s) in the quark Yukawa matrices affect both the CP-violating and CP-conserving parameters of the quark mixing matrix. Our analysis does not assume that the KM mechanism is the only source of CPV.
- [3] Our methods can also be applied to the leptonic sector, as we shall discuss elsewhere.

- [4] The η_{jk} term allows for the possibility of making the rephased element real and negative; this will not affect the counting of unremovable phases.
- [5] A. Kusenko and R. Shrock, ITP-SB-93-62, to appear.
- [6] Although our theorems are general, we note that much recent work has been done studying models for fermion masses and quark mixing in supersymmetric grand unified theories (SGUT's) with evolution equations given by the minimal supersymmetric standard model (MSSM); see, e.g., Refs. [7]-[11] and references to earlier work therein. These typically start with $|Y_{jk}^{(f)}| = |Y_{kj}^{(f)}|$, which can be achieved naturally in SO(10) and minimizes parameters. Discrete symmetries which could produce zeroes can arise from an underlying string theory, but string theories do not generically yield (simple) grand unified groups or symmetric Yukawa matrices.
- [7] H. Arason, D. Castaño, B. Keszthelyi, S. Mikaelian, E. Piard, P. Ramond, and B. Wright Phys. Rev. Lett.**67**, 2933 (1991); Phys. Rev. D**46**, 3945 (1992); H. Arason, D. Castaño, P. Ramond, and E. Piard, Phys. Rev. D**47**, 232 (1993).
- [8] S. Dimopoulos, L. Hall, and S. Raby, Phys. Rev. D**45**, 4195 (1992); G. Anderson, S. Raby, S. Dimopoulos, L. Hall and G. Starkman, LBL-33531 (hep-ph/9308333).
- [9] V. Barger, M. Berger, and P. Ohmann, Phys. Rev. D**47**, 1093 (1992).
- [10] P. Ramond, R. Roberts, and G. G. Ross, RAL-93-010/UFIFT-93-010 (hep-ph/9303320) give a comprehensive list of five viable models M_j , $j = 1 - 5$ of quark Yukawa matrices with $|Y_{jk}^{(f)}| = |Y_{kj}^{(f)}|$, $f = u, d$. Denote the generalization to nonsymmetric $Y^{(f)}$ as M'_j ; this does not affect the counting or allowed placement of the phases. Models 1 – 3 in the text are M'_j , $j = 1 - 3$.
- [11] A. Kusenko and R. Shrock, ITP-SB-93-37 (hep-ph/9307344) studied a SGUT model for quark and lepton Yukawa matrices which gives a new realization of the Georgi-Jarlskog mass relation $m_d = 3m_e$, $m_s = m_\mu/3$, $m_b = m_\tau$; the quark part is M_1 with $|B_{22}| = |B_{23}|$.

- [12] In the model of Ref. [11], the unremovable phase in the set S_1 did not significantly help the fit to data and hence was taken to be zero to minimize the number of parameters.
- [13] These results differ with statements in Ref. [10] that the $Y^{(f)}$ can be made hermitian and with the counting of phases given there. For the M'_j , $j = 1 - 5$, we find the values of N_p are respectively 2, 3, 2, 2, 2, not 1, 2, 1, 1, 1 as claimed. Further, $Y^{(d)}$ cannot in general be made hermitian (nor can the submatrix S_1 be made real or hermitian) in M'_j , $j = 1, 2, 3$, and $Y^{(u)}$ cannot be made real (or hermitian) in M'_4 or M'_5 . We find the following invariants for the other two models: M'_4 : $(P_{22,33}^{(u)}, Q_{12,22})$, M'_5 : $(P_{22,33}^{(u)}, Q_{13,22})$. We have communicated our results to P. Ramond and G. G. Ross and thank them for discussions.
- [14] H. Fritzsch, Phys. Lett. B**73**, 317 (1978); Nucl. Phys. B**155**, 182 (1979).
- [15] G. Giudice, Mod. Phys. Lett. A **7**, 2429 (1992).
- [16] The special case with $|Y_{jk}^{(f)}| = |Y_{kj}^{(f)}|$, $f = u, d$ has been studied in Refs. [7]-[9].